3 (Sem–6/CBCS) MAT HC 1

2022

MATHEMATICS

(Honours)

Paper : MAT–HC–6016

(Complex Analysis)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any seven*** questions from the following : $1 \times 7 = 7$

   (a) If c is any nth root of unity other than unity itself, then value of $1 + c + c^2 + \cdots + c^{n-1}$ is

   (i) $2n\pi$

   (ii) 0

   (iii) -1

   (iv) None of the above

   (Choose the correct answer)

(b) The square roots of $2i$ is

(i) $\pm(1+i)$

(ii) $\pm(1-i)$

(iii) $\pm\frac{1}{\sqrt{2}}\left(1-i\sqrt{2}\right)$

(iv) None of the above

(Choose the correct answer)

(c) A composition of continuous function is

(i) discontinuous

(ii) itself continuous

(iii) pointwise continuous

(iv) None of the above

(Choose the correct answer)

(d) The value of $Log(-ei)$ is

(i) $\frac{\pi}{2}-i$

(ii) i

(iii) $1-\frac{\pi}{2}i$

(iv) None of the above

(Choose the correct answer)

(e) The power expression of $\cos z$ is

(i) $\frac{e^{z}+e^{-z}}{2}$

(ii) $\frac{e^{iz}+e^{-iz}}{2}$

(iii) $\frac{e^{iz}+e^{-iz}}{2i}$

(iv) None of the above

(Choose the correct answer)

(f) The Cauchy-Riemann equation for analytic function $f(z)=u+iv$ is

(i) $u_x=v_y$, $u_y=-v_x$

(ii) $u_x=-v_y$, $u_y=v_x$

(iii) $u_{xx}+v_{yy}=0$

(iv) None of the above

(Choose the correct answer)

(g) If $w(t)=u(t)+iv(t)$, then $\frac{d}{dt}[w(t)]^2$ is equal to

(i) $2[u(t)+iv(t)]$

(ii) $2w'(t)$

(iii) $2w(t)w'(t)$

(iv) None of the above

(Choose the correct answer)

(h) What is Laplace's equation ?

(i) What is extended complex plane ?

(j) What is Jordan arc ?

2. Answer **any four** questions from the following : 2×4=8

(a) Write principal value of $arg\left(\frac{i}{-1-i}\right)$.

(b) If $f(z)=x^2+y^2-2y+i(2x-2xy)$, where $z=x+iy$, then write $f(z)$ in terms of z.

(c) Use definition to show that $\lim_{z\to z_0} \bar{z}=\bar{z}_0$.

(d) Find the singular point of

$$f(z)=\frac{z^2+3}{(z+1)(z^2+5)}.$$

(e) If $f'(z)=0$ everywhere in a domain D, then prove that $f(z)$ must be constant throughout D.

(f) Evaluate $f'(z)$ from definition, where $f(z)=\frac{1}{z}$.

(g) If $f(z)=\frac{z}{\bar{z}}$, find $\lim_{z\to 0} f(z)$, if it exists.

(h) Write the function $f(z)=z+\frac{1}{z}\ (z\neq 0)$ in the form $f(z)=u(r,\theta)+iv(r,\theta)$.

3. Answer **any three** questions from the following : 5×3=15

(a) If z_1 and z_2 are complex numbers, then show that $\sin(z_1+z_2)=\sin z_1\cos z_2+\cos z_1\sin z_2$.

(b) Show that exp. $(2\pm 3\pi i)=-e^2$.

(c) Sketch the set $|z-2+i|\leq 1$ and determine its domain.

(d) Let C be the arc of the circle $|z|=2$ from $z=2$ to $z=2i$, that lies in the 1st quadrant, then show that

$$\left|\int_C \frac{z-2}{z^2+1}\,dz\right|\leq\frac{4\pi}{15}$$

(e) Evaluate $\int_C \frac{dz}{z}$, where C is the top half of the circle $|z|=1$ from $z=1$ to $z=-1$.

(f) If $f(z)=e^z$, then show that it is an analytic function.

(g) If $f(z)=\frac{z+2}{z}$ and C is the semi circle $z=2e^{i\theta}$, $(0\le\theta\le\pi)$, then evaluate $\int_C f(z)\,dz$.

(h) Find all values of z such that $e^z=-2$.

4. Answer **any three** questions from the following: 10×3=30

(a) State and prove Cauchy-Riemann equations of an analytic function in polar form.

(b) Suppose that $f(z)=u(x,y)+iv(x,y)$, $(z=x+iy)$ and $z_0=x_0+iy_0$, $w_0=u_0+iv_0$, then prove that if $\lim_{(x,y)\to(x_0,y_0)} u(x,y)=u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y)=v_0$ then $\lim_{z\to z_0} f(z)=w_0$ and conversely.

(c) If the function $f(z)=u(x,y)+iv(x,y)$ is defined by means of the equation

$$f(z)=\begin{cases}\frac{\bar{z}^e}{z}, & \text{when } z\neq 0\\ 0, & \text{when } z=0,\end{cases}$$

then prove that its real and imaginary parts satisfies Cauchy-Riemann equations at $z=0$. Also show that $f'(0)$ fails to exist.

(d) If the function $f(z)=u(x,y)+iv(x,y)$ and its conjugate $\bar{f}(z)=u(x,y)-iv(x,y)$ are both analytic in a domain D, then show that $f(z)$ must be constant throughout D.

(e) If f be analytic everywhere inside and on a simply closed contour C, taken in the positive sense and z_0 is any point interior to C, then prove that

$$f(z_0)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz.$$

(f) State and prove Liouville's theorem.

(g) Suppose that a function f is analytic throughout a disc $|z - z_0| < R_0$ centred at z_0 and with radius R_0. Then prove that $f(z)$ has the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad (|z - z_0| < R_0)$$

$$\text{where } a_n = \frac{f^n(z_0)}{\lfloor n}, \quad (n = 0, 1, 2, \ldots\ldots)$$

(h) State and prove Laurent's theorem.

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